Estimating time trends in Gumbel-distributed data by means of generalized linear models

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Received 4 September 2001; revised 19 February 2002; accepted 19 February 2002; published 23 July 2002.

[1] This paper shows how Gumbel-distributed data can be related to explanatory variables by using generalized linear models (GLMs) fitted by using a modified form of the iteratively weighted least squares algorithm (IWLS). Typical applications include (1) testing for trend in annual flood data, as a possible consequence of changing land cover or other factors; (2) testing for trend in annual maximum rainfall intensities of different durations, as a possible consequence of climate change; and (3) testing how annual maximum rainfall intensity is related to weather conditions at the times that annual maximum intensities were recorded. Given a first estimate of the Gumbel scale parameter α, the coefficients β of explanatory variables $\boldsymbol{x}$ are estimated by casting the model in GLM form, and the scale parameter α is updated by solution of relevant maximum likelihood equation for this parameter. The parameters α, β can be readily estimated using currently available statistical software for fitting GLMs, which can also be used to test the significance of trends in annual flood data for which the Gumbel distribution is a plausible hypothesis. A plotting procedure to indicate departures from the Gumbel hypothesis is also given. The proposed procedure avoids the illogicality in which, when a trend in flood data is suspected, it is tested either by linear regression methods that assume Normally distributed residuals, or by nonparametric methods, both of which discard the Gumbel hypothesis. Simulated samples from Gumbel distributions were used to compare estimates of linear trend obtained by (1) the GLM procedure and (2) straightforward use of a Newton-Raphson procedure to locate the maximum of the likelihood surface; the GLM procedure converged more rapidly and was far less subject to numerical instabilities. Simulated samples from Gumbel distributions were also used to compare estimates of a linear trend coefficient β given by the GLM procedure, with estimates of β obtained by simple linear regression (LR). The variance of the distribution of GLM estimates of β was less than the variance of the distribution of LR estimates, while comparison of the powers of the two tests showed that GLM was more powerful than LR at detecting the existence of small trends, although for large linear trends there was little to choose between the two methods. *INDEX TERMS:* 1821 Hydrology: Floods; 1860 Hydrology: Runoff and streamflow; 1630 Global Change: Impact phenomena; 1694 Global Change: Instruments and techniques; *KEYWORDS:* Gumbel, trend, power

1. Introduction

[2] In regions where climate or land use is changing, the common assumption that hydrologic time series are stationary must be called into question. Annual flood records (typically, sequences of annual maximum daily flow) may well show changes over the years as a consequence of more rapid runoff associated with increased urbanization, changes to agricultural practices, or deforestation within the watershed [*Bruijnzeel*, 1990, 1996; *Sahin and Hall*, 1996]. In the absence of time trends, the Gumbel distribution

$$f(y;\alpha,m) = \alpha\exp[-\alpha(y-m) - \exp\{-\alpha(y-m)\}] \quad -\infty < y < \infty \tag{1a}$$

with cumulative probability

$$F(y;\alpha,m) = \exp[-\exp\{-\alpha(y-m)\}] \tag{1b}$$

has been widely used for estimating the magnitudes of floods with different ("*T*-year") return periods, because of the ease with which its quantiles can be calculated, the flood with return period T years being the quantile corresponding to cumulative probability $1 - 1/T$. As is well known [e.g., *Stedinger et al.*, 1993], the location and scale parameters in (1) are m and α, while the mean, standard deviation and skewness are $m+\gamma/\alpha$, $\pi/(\alpha\sqrt{6})$ and 1.1396 respectively, where γ is Euler's constant, $\gamma = 0.577215\ldots$ and $\pi = 3.14159\ldots$. There is now a very wide literature on the use of the Gumbel distribution in flood frequency studies [e.g., *Natural Environment Research Council*, 1975; *Stedinger et al.*, 1993].

[3] The Gumbel distribution is also extensively used in the study of extreme rainfall intensities. Where annual maximum

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intensities are recorded over durations D_1, D_2, ... D_k, a Gumbel distribution is fitted to data from each duration [e.g., *Buishand*, 1993], and the quantiles of the k distributions are plotted as intensity-duration-frequency (IDF) curves for subsequent use in urban planning. IDF curves are typically used for flood peak estimation in techniques such as the rational method [*Pilgrim and Cordery*, 1993].

[4] Use of the Gumbel (or indeed any other) distribution to assess the frequency of extreme events from a historic record of annual maxima, depends critically on the assumption that the distribution from which the data are a sample remains constant over time. If the historic flow record shows a time trend, the concept of a flood with return period T years becomes meaningless. It may still be possible to make statements about future flood characteristics using rainfall-runoff models that incorporate assumptions about future weather patterns (where nonstationarity is believed to be the result of climate change), and/or assumptions about future land use changes (where nonstationarity may be caused by urbanization, deforestation, or change in agricultural practices). But the essential point is that frequency statements can no longer be made from a statistical analysis of historic records alone, since these will no longer be a guide to future conditions.

[5] The Intergovernmental Panel on Climate Change (IPCC) predicts changes, both in frequency and severity of intense rainfall, as a consequence of global warming; Table SPM 1 of the *Intergovernmental Panel on Climate Change* (*IPCC*) [2001] "Impact, Adaptation and Vulnerability" foresees "more intense precipitation events, very likely over many areas", with "increased susceptibility to flooding". Detection of time trends in records of annual floods and annual maximum rainfall intensity will become of increasing importance if the IPCC predictions come to pass. Even with existing records, regional increases in the amount and intensity of North American rainfall have been reported by *Vinnikov et al.* [1990], *Guttman et al.* [1992], *Groisman and Easterling* [1994], and *Karl and Knight* [1998]. In analyses of flood flows, *Changnon and Kunkel* [1995] found significant upward trends in floods in the northern Midwest during the period 1921–1985, while *Olsen et al.* [1999] found large and statistically significant upward trends in flood flows over the last 100 years in the Upper Mississippi and Missouri rivers. However, when the effects of spatial correlation between flood flows were taken into account, the significance of such trends in flood flows was greatly diminished [*Douglas et al.*, 2000]. The assessment of time trends in flood flows therefore requires great care and is likely to become increasingly important if and when effects of climate change become more apparent.

[6] Present procedures for detecting tends in annual flood flows or maximum rainfall intensities are generally of two types: (1) linear regression analysis of the annual maxima on year number, for which the residuals, and hence the annual maxima themselves, are assumed to follow a Normal distribution, possibly with time-varying mean [*Hirsch et al.*, 1993; *Salas*, 1993]; (2) nonparametric tests such as the Mann-Kendall trend test [*Salas*, 1993] which, being nonparametric and rank-based, makes no assumption concerning the distribution of the data. Neither procedure utilizes the fact that the annual maxima can be expected to follow a Gumbel (or perhaps some other extreme value) distribution when no significant trend is detected. This paper considers the case where the working hypothesis is that data follow a Gumbel distribution, in which a time trend may or may not exist, and for which a test of trend is required. The objectives of the present paper are to show (1) how the iteratively weighted least squares (IWLS) algorithm [*Green*, 1984; *McCullagh and Nelder*, 1989, pp. 40–43] provides a test for the existence of a time trend in data that can plausibly assumed to be Gumbel-distributed, whether or not trend exists; (2) how the Gumbel assumption can be checked graphically when a significant time trend is detected; (3) how the test for trend based on the Gumbel hypothesis compares with the test for trend using simple regression methods. The emphasis of the paper is on testing for trend in the mean of a Gumbel distribution, but the same procedure can in principle be adapted to test for trend in the Gumbel scale parameter α. Where a significant time trend in the Gumbel mean is detected, however, there will usually be little point in taking the further step of testing for trend in α. Nonstationarity in the mean having been demonstrated, calculation of quantiles, T-year floods (and IDF curves in the case of rainfall intensity data) are no longer relevant and the possibility of nonstationarity in α becomes of marginal interest.

2. Gumbel Distribution With Time-Variant Mean

[7] The present paper uses the Gumbel distribution with time-variant mean, which in its simplest form is obtained by replacing the distribution in (1) by

$$f(y;\alpha,m,\beta) = \alpha\exp[-\alpha(y-m-\beta t) - \exp\{-\alpha(y-m-\beta t)\}] \tag{2}$$

where t is a time variable, so that (2) reduces to (1) when $\beta = 0$. The mean of the distribution in (2), for a given time t, is $(m + \gamma/\alpha) + \beta t$, which as t varies is a line with slope β and intercept $(m + \gamma/\alpha)$. More elaborate time trends can obviously be explored by substituting the single-parameter trend βt by the k-parameter trend $\beta^T\boldsymbol{x}$, where $\beta = [\beta_1, \beta_2,\ldots \beta_k]^T$, $\boldsymbol{x} = [f_1(t), f_2(t),\ldots. f_k(t)]$ and the $f_i(t)$ are functions of time, typically polynomials or harmonic functions. Even where trends over time are absent, it may still be of interest to explore the relation between a Gumbel variate and one or more explanatory variables. For example, in an analysis of annual maximum rainfall intensity, it may be of interest to explore whether the observed annual maxima are related to wind direction and velocity, or to time of day, at which the maximum intensities occurred. We illustrate maximum likelihood tests for the hypothesis H_0: $\beta = 0$, and show how the powerful family of GLMs can be used to extend the calculation in a straightforward manner when k, the number of explanatory variables, is larger.

3. Testing the Hypothesis That No Trend Exists Against the Alternative of a Linear Trend

[8] We assume that a sequence of N data values is available for making inferences about time trends, and for simplicity, assume only one trend parameter, although any number of trend parameters could be tested (subject to

limitations imposed by the record length). Thus the data value y_i in the ith year is taken to be an observation of a random variable Y_i having a Gumbel distribution with location parameter $E[Y_i | t_i] = (m + \gamma/\alpha) + \beta t_i$ and such that Y_i and Y_k are independent for $i \neq k$. The usual assumption is therefore made that annual floods are independent from year to year. The log likelihood function, regarded as a function of the three parameters α, m, β for response variable $\boldsymbol{y}$ given explanatory observations $\boldsymbol{t}$ is

$$\log_e L = N \log_e \alpha - N\alpha\bar{y} + N\alpha m + N\alpha\beta\bar{t} - \sum_{i=1}^{N} \exp(-\alpha[y_i - m - \beta t_i]). \quad (3)$$

where $\bar{y}, \bar{t}$ are means over the N data values and N times respectively. With just three parameters, this function can be maximized by the Newton-Raphson method, in which each of the three equations $\partial \log_e L/\partial m = 0$, $\partial \log_e L/\partial \alpha = 0$, $\partial \log_e L/\partial \beta = 0$ is expanded as a Taylor series in Δm, $\Delta\alpha$, $\Delta\beta$, second- and higher-order terms are discarded, and the linearized equations are solved for the corrections Δm, $\Delta\alpha$, $\Delta\beta$ to be added to the starting values m_0, α_0, β_0, the whole calculation being iterated until a convergence criterion is satisfied. The Newton-Raphson method can readily be extended to the case where there is a vector β of trend parameters, but it has two disadvantages: (1) as a computational procedure, it fails to take account of the linear structure in the parameters m, β. The procedure based on a GLM fitted by the IWLS algorithm, described later in this paper, utilizes this linear structure, and is therefore more efficient computationally. (2) Although it estimates the trend parameters β, it does not of itself provide tests of hypotheses that some or all of the trend coefficients are zero. Thus with the single trend parameter shown in equation (2), a test of the hypothesis H_0: $\beta = 0$ requires a second Newton-Raphson calculation by maximizing equation (3) in which β has been set equal to zero. The statistic $-2\ \{\log_e L(H_0) - \log_e L(H_1)\}$ is then asymptotically distributed as χ^2 with one degree of freedom. Each time that a hypothesis must be tested concerning inclusion or exclusion of a trend parameter, additional computer programming is required, both to calculate the maximized likelihood function, and to calculate the corresponding likelihood-ratio statistic. While this presents no intrinsic difficulty, most of this additional calculation can be avoided by using readily available GLM procedures to fit the trend parameters β and to test them for statistical significance.

4. Use of GLMs To Fit Trend Parameters and Test Their Statistical Significance

[9] GLMs constitute an extension to multiple regression models, a major difference being that the data are no longer required to be normally distributed. In common GLM usage, the data may be a sample from any distribution belonging to the exponential family defined as follows. Denoting by y an observation of the random variable Y, then the distribution of Y belongs to the exponential family if it can be written in the form [*McCullagh and Nelder*, 1989]

$$f_Y(y; \theta, \phi) = \exp\{(y\theta - b(\theta))/a(\phi) + c(y, \phi)\} \quad (4)$$

for some specific functions $a(\ )$, $b(\ )$, $c(\ )$. It can be shown that the Poisson, binomial, gamma and inverse Gaussian, as well as the normal, all have distributions that can be written (after some algebraic manipulation) in the form equation (4) [*McCullagh and Nelder*, 1989]. For all distributions in the exponential family, parameters are estimated by "sufficient" statistics that utilize all information on them that is provided by the data, whatever the sample size. If ϕ is known, equation (4) defines an exponential family model with canonical parameter θ; if ϕ is unknown, it may or may not be an exponential family model. In exponential family models, the maximum likelihood estimator of θ is a sufficient statistic for this parameter; the mean and variance of the random variable Y are related to $b(\theta)$ and $a(\phi)$ by

$$E[Y] = \mu = b'(\theta) \quad (5a)$$

$$\text{var}[Y] = b''(\theta)a(\phi) \quad (5b)$$

where b';(θ) and $b''(\theta)$ are the first and second derivatives of $b(\theta)$ with respect to θ. GLMs also extend multiple regression theory by allowing the mean $E[Y_i] = \mu_i$ of the ith data value to be related to explanatory variables $\boldsymbol{x}$ not only by means of $E[Y_i] = \boldsymbol{x}_i^T \beta$, as in ordinary multiple regression, but also by the more general form $g(\mu_i) = \boldsymbol{x}_i^T \beta$ where $g(.)$ is termed the link function. Goodness of GLM fit is measured by the Deviance, denoted by D, which is related to the difference between two log likelihoods, one for the model whose fit is to be assessed, and the other for a "full" model in which each observation is regarded as sampled from a distribution with different mean value. Thus the algebraic expression for the deviance D differs for each member of the exponential family. For the special case in which the distribution of Y_i is Normal, and the link function is the identity function $g(\mu_i) = \mu_i = \boldsymbol{x}_i^T \beta$, the deviance becomes simply the residual sum of squares (RSS) of the multiple regression. In fact GLMs are fitted by minimizing the Deviance statistic, analogous to minimizing the RSS when fitting a multiple regression.

[10] We now come to the relevance of this theory to fitting the Gumbel distribution in which the parameter m in equation (1) is replaced by $\boldsymbol{x}^T \beta$ where $\boldsymbol{x}$ is a vector of explanatory variables (in the last section, $\boldsymbol{x}$ was a vector with only one explanatory variable: namely t). Suppose for the moment that the scale parameter α is known. Then from the N data values y_i $\{i = 1\ldots N\}$ a new sequence z_i can be formed from the transformation $z = -\exp(-\alpha y)$. The Gumbel distribution equation (1) can then be written in the exponential family form equation (4) as

$$f_z(z) = \exp[z\theta + \log_e \theta + \log_e(-\alpha z)] \quad (6)$$

where $z = -\exp(-\alpha y)$, $\theta = \exp(\alpha m)$. As the original random variable Y ranges from $-\infty$ to ∞, the new variable Z ranges from $-\infty$ to zero, and so is negative. Comparing equation (6) with the required form equation (4), we see that $a(\phi) = 1$, $b(\theta) = -\log_e(\theta)$, and $c(y, \phi) = \log_e(-\alpha z)$, recalling that α is assumed known for the present. From equation (5a), $\mu = b'(\theta) = -1/\theta$, and from equation (5b), $\text{var}[Z] = b''(\theta)\ a(\phi) = 1/\theta^2$ (since $a(\phi) = 1$). From the definition of the deviance [*McCullagh and Nelder*, 1989, p. 33],

$$D = 2\sum_{i=1}^{N}\left\{z_i\left(\tilde{\theta}_i - \hat{\theta}_i\right) - b(\tilde{\theta}_i) + b\left(\hat{\theta}_i\right)\right\} \quad (7)$$

where $\tilde{\theta}_i$ and $\hat{\theta}_i$ are estimates of the parameter θ under the full and current models respectively. Since, $\theta = 1/\mu$ we have $\tilde{\theta}_i = 1/z_i$ and $\hat{\theta}_i = -1/f_i$, where f_i is the fitted value for the ith observation z_i under the current model, calculated according to the IWLS algorithm [*McCullagh and Nelder*, 1989, p. 40–43].

[11] The above development assumes that the scale parameter α is known, so that the new variable $z = -\exp(-\alpha y)$ can be calculated from the original data values y_i. In practice, α is not known, and must be estimated. One solution is to estimate it from the equation $\partial\log_e L/\partial\alpha = 0$, with $\hat{m}$ and $\hat{\beta}$ temporarily held fixed. The following algorithm provides for the iterative estimation of α and the parameters β that are the coefficients of the explanatory variables $\boldsymbol{x}$:

1. Estimate α from the data, ignoring trend, from the usual method of moments estimator $\tilde{\alpha} = \pi/\left(\tilde{\sigma}\sqrt{6}\right)$, where $\tilde{\sigma}$ is the standard deviation of the y_i $\{i = 1\ldots N\}$.
2. Use this estimate of α to obtain estimates of m and the coefficients β of the explanatory variables, using the iteratively weighted least squares algorithm and the model expressed in GLM terms, with a log link function.
3. Use these estimates of m and β to solve the following equation for α:

$$N/\alpha - \sum_{i=1}^{N}\left[y_i - m - \beta^T\mathbf{x}_i\right] + \sum_{i=1}^{N}\left(y_i - m - \beta^T\mathbf{x}_i\right)\exp\left[-\alpha\left(y_i - m - \beta^T\mathbf{x}_i\right)\right] = 0. \quad (8)$$

Equation (8) is the equation $\partial\log_e L/\partial\alpha = 0$ yielding the maximum likelihood estimate of the Gumbel scale parameter α.

4. With this new value of α, return to step 2 to calculate revised estimates $\hat{m}, \hat{\beta}$. Iterate Steps 2 and 3 until convergence.

[12] This computational procedure is illustrated in section 5.

5. Numerical Example: Testing for Trend in Two Annual Flood Series at Sites on the Rio Jacuí, Southern Brazil

[13] Table 1 shows the annual maximum floods (more exactly, annual maximum mean daily discharges, with mean daily discharge computed as the mean of two discharges corresponding to gauge board readings at 0700 and 1900 hours) for two gauging sites, Espumoso and Passo Bela Vista, on the Rio Jacuí, in southern Brazil. Over the past three decades, deforestation in the Jacuí drainage basin has been extensive, with indigenous forest replaced by intensive agriculture. Table 2 shows the convergence achieved by the iteratively weighted least squares algorithm for the model containing a single trend parameter as shown in equation (2).

Table 1. Annual Maximum Mean Daily Flows ("Annual Floods"), 1940–1993, for Two Stations (Espumoso and Passo Bela Vista) on the River Jacuí, Southern Brazil[a]

Year	Espumoso	Bela Vista
1940	*	900
1941	312	1678
1942	590	866
1943	248	455
1944	670	761
1945	365	510
1946	770	674
1947	465	548
1948	545	621
1949	315	415
1950	115	1495
1951	232	346
1952	260	354
1953	655	985
1954	675	1186
1955	455	680
1956	1020	1525
1957	700	785
1958	570	1062
1959	853	1074
1960	395	607
1961	926	1124
1962	99	132
1963	680	868
1964	121	615
1965	976	1068
1966	916	1062
1967	921	1225
1968	191	313
1969	187	278
1970	377	619
1971	128	1121
1972	582	1498
1973	744	1062
1974	710	1309
1975	520	674
1976	672	1017
1977	645	796
1978	655	692
1979	918	1202
1980	512	785
1981	255	391
1982	1126	1501
1983	1386	1933
1984	1394	1726
1985	600	796
1986	950	1239
1987	731	1009
1988	700	931
1989	1407	1593
1990	1284	1423
1991	165	293
1992	1496	1790
1993	809	905

[a] Units are $m^3\ s^{-1}$. Asterisk denotes missing value.

For Espumoso, the fitted trend is $(\hat{m} + \gamma/\hat{\alpha}) + \hat{\beta}t = 484.72 + 5.96\ t$, giving a likelihood ratio criterion of $\chi^2 = 15.842$ on one degree of freedom; for Passo Bela Vista, the fitted trend is $804.82 + 5.26\ t$, with $\chi^2 = 2.382$ on one degree of freedom. Tabulated values of χ^2 at the conventional 5%, 1% and 0.1% significance levels are 3.841, 6.635, and 10.827, respectively, showing highly significant time trend at Espumoso, and no significant time trend at Passo Bela

Table 2. Annual Flood Data for Espumoso and Passo Bela Vista: Convergence Achieved by the Iteratively Weighted Least Squares Algorithm

	Espumoso			Passo Bela Vista		
Iteration	$\hat{\alpha}$	$\hat{m}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{m}$	$\hat{\beta}$
1	0.003419	314.17	5.7819	0.002715	581.64	4.9402
2	0.003403	315.20	5.9314	0.002690	588.81	5.2231
3	0.003401	315.33	5.9494	0.002687	589.57	5.2528
4	0.003401	315.35	5.9516	0.002687	589.66	5.2562
ML	0.003401	315	5.96	0.002687	590	5.26

Vista. Both fits are equal to those obtained using the Newton-Raphson procedure, but are achieved with fewer iterations. Although iteration was necessary to solve the equation (8) for α after each iteration for m and β, this calculation usually converged at the second iteration.

[14] It is a straightforward calculation to estimate the coefficients of additional explanatory variables. Suppose that the significance of a quadratic trend component is required for the Espumoso data, so that the explanatory variables are t and t^2. As with the more familiar multiple regression, we fit t first, and then test whether further inclusion of the t^2 variable improves the fitted trend. It is found that the reduction in the Deviance statistic due to inclusion of t^2 is far from significant. Since a log-link function is used, as explained above, the fitted model is $\log_e(\mu(z_i)) = -1.40 + 0.011t_i - 0.00055t_i^2$, where $\mu(z_i)$ is the mean value of the transformed variable $z = \exp(-\alpha y)$ evaluated at $t = t_i$. Step 2 of the above algorithm converges to give α estimated as 0.0003176. Since $\mu(z_i) = \exp[-1.40 + 0.011t_i - 0.00055t_i^2]$ and $\mu(z_i) \approx \exp(-\alpha\mu(y_i))$, back-transformation gives the fitted model $E[Y|t_i] = 409.19 - 3.23\ t + 0.16\ t^2$. Coefficients of additional powers t^3, t^4... can be fitted if required. All of this calculation is easily effected by standard statistical software; results given in this paper were obtained using the commercially available Genstat, but SAS, GLIM and other programs can also be used. The inclusion of new parameters in the vector β, and the omission from it of others, is as straightforward as in multiple regression; indeed in Genstat, the same two directives (MODEL and FIT) can used either to fit GLMs, or to fit regressions.

[15] It is of interest to compare the estimates of $\hat{\beta}$ obtained from the modified Gumbel model equation (2) with the estimates $\hat{\beta}$ obtained from linear regressions of annual flood on year of occurrence at the two sites. These are very different: for Espumoso and Passo Bela Vista respectively, the values of $\hat{\beta}$ are 10.63 ± 2.91 and 8.31 ± 3.65 $m^3\ s^{-1}\ yr^{-1}$, statistically significant at the 0.1% and 5% levels respectively. However the coefficients of determination are both small: 19.1% and 7.3%. Using the notation given by *Douglas et al.* [2000] for a Mann-Kendall nonparametric test for trend, the test statistic Z, whose distribution is well approximated by $N(0, 1)$ for these sample sizes, gave $Z = 2.762$ for Espumoso, significant at the 1% level, and $Z = 1.865$ for Passo Bela Vista, less than the value for significance at the 5% level.

5.1. Extension of the Method to Test for Trend in the Gumbel Scale Parameter α

[16] If the GLM fit shows that some or all of the trend parameters β are statistically significant, this is sufficient to demonstrate that the usual analytical procedures (estimation of floods with return period T years; or calculation of IDF curves in the case of rainfall intensity) are no longer valid, and that other methods are required to explore frequencies of occurrence, because past behavior is then at best an unreliable guide to future behavior. However a question arises concerning whether it is possible to test for time trend in the Gumbel scale parameter α as well as in the Gumbel mean. Provided that a particular form for the trend in α can be specified (such as replacing the α in equation (2) by $\alpha_0 + \alpha_1 t$, with the parameters α_0, α_1 to be estimated) it is straightforward in principle to extend the GLM fitting procedure to estimate α_0 and α_1. Step 1 of the computation proposed above would set the initial value of α_0 to $\pi/(\tilde{\sigma}\sqrt{6})$, where $\tilde{\sigma}$ is the standard deviation of the $y_i\{i = 1...N\}$; the initial value of α_1 would be set to zero. Step 2 would proceed in the usual way to give estimates of m and β, but at step 3 two equations now need to be solved iteratively: namely the maximum likelihood equations $\partial\log_e L/\partial\alpha_0 = 0$ and $\partial\log_e L/\partial\alpha_1 = 0$. The estimates of α_0 and α_1 are then used to calculate the time variation $\alpha_0 + \alpha_1 t$ in the scale parameter, and hence the new variable $z_i = = -\exp(-[\alpha_0 + \alpha_1 t_i]y_i)$ for entry into the GLM procedure. Further work is necessary to establish whether, and under what conditions, this procedure would fail to converge.

5.2. Graphical Validation of the Gumbel Hypothesis

[17] Where the Gumbel distribution with time-variant mean has been taken as an initial working hypothesis, validation of this hypothesis can be achieved by analysis of the residuals about a fitted trend line. In the notation used earlier in this paper, the residuals are $y_i - f_i$, where f_i are fitted values. These are placed in ascending order to give $y_{(i)} - f_{(i)}$, $i = 1, 2 \ldots N$. If these follow a Gumbel distribution they should show high correlation with the N quantiles Q_i from the standard Gumbel distribution $F(u) = \exp(-\exp(-u))$. A plot of the Q_i against the ordered residuals $y_{(i)} - f_{(i)}$ should then appear as a straight line (a "*Q-Q* plot"). Figures 1 and 2 show histograms of the residuals, and *Q-Q* plots for the annual flood data from Espumoso and Passo Bela Vista. The N quantiles from the standard Gumbel distribution were calculated using the Hazen formula, and found by solving the relation $F(u_i) = (i - 0.5)/N$ for the u_i which are plotted on the vertical axis of the *Q-Q* plots in Figures 1 and 2. Other choices of the u_i are also possible [*Stedinger et al.*, 1993, Table 18.3.1]. The figures show *Q-Q* plots that are linear over much of the range of residuals, with some notable discrepancies among the larger ones. However, the two largest residuals 808.8 and 810.8 at Espumoso, where significant trend was detected, occurred in the two years 1983 and 1984, a period of very strong El Niño conditions as Table 1 shows.

6. Computational Considerations

[18] The question can be asked "why go to the trouble of testing for the existence of trend by casting the problem in the form of a GLM, and using the IWLS algorithm, instead of using a Newton-Raphson procedure to locate the likelihood maximum directly?" The short answer to this question is the Newton-Raphson procedure often fails, particularly when several trend (or other) parameters β are to be estimated. To illustrate what happens, 50 simulated samples $\{y\}$, each of length $N = 50$, were drawn from a standard Gumbel distri-

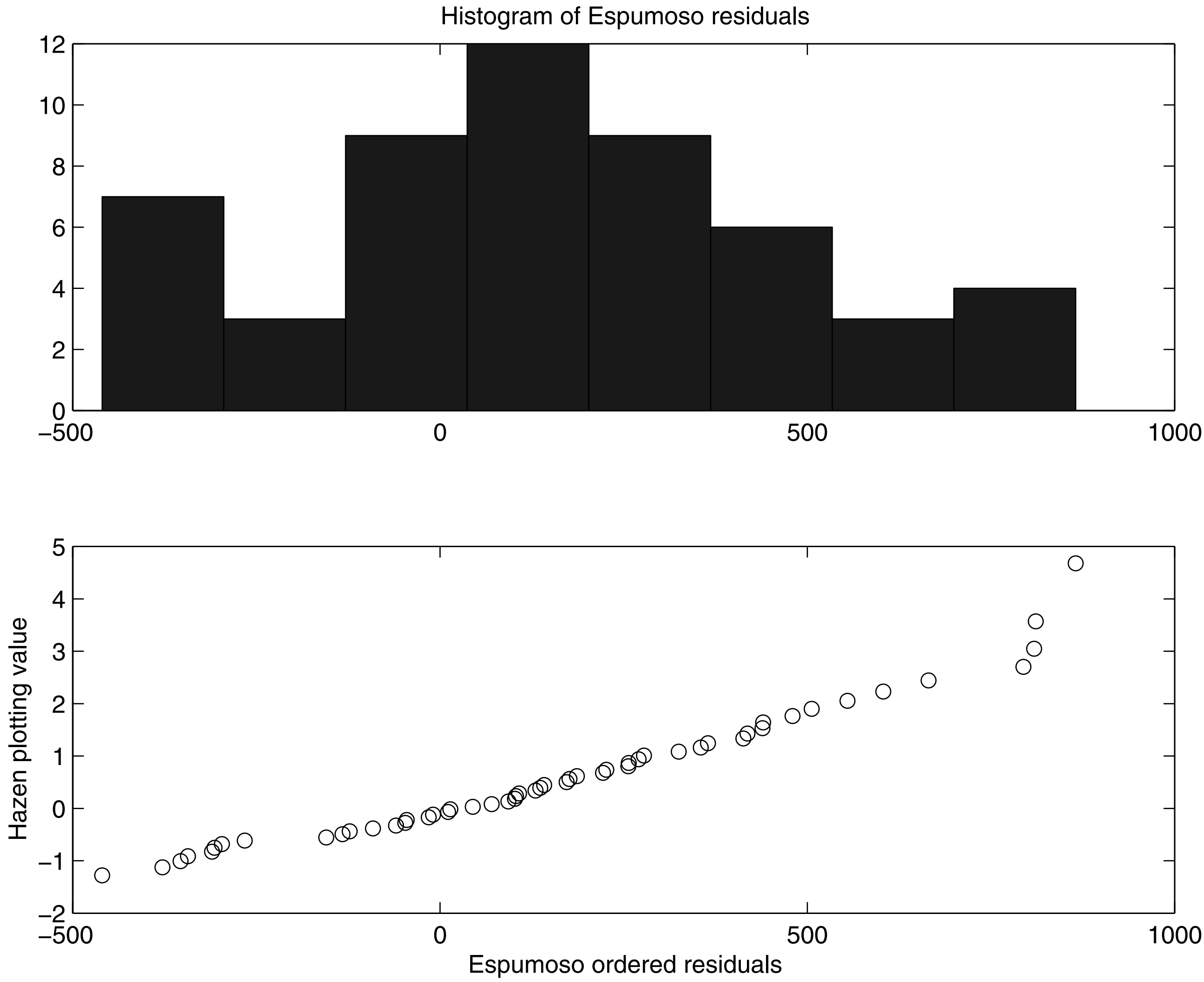

Figure 1. Histogram of Espumoso residuals and *Q-Q* plot for consistency with the Gumbel hypothesis.

bution with $m = 0$, $\alpha = 1$; each simulated sample was taken as a set of observations of a dependent variable, and five independent variables x_1, x_2, x_3, x_4, x_5, all correlated with the $\{y\}$, were constructed by adding uniformly distributed (0, 1) "errors" to the set $\{y\}$. Thus the "dependent" variable Y was taken to have the distribution $f_Y(y) = \alpha \exp [-\alpha(y - m - \beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3 - \beta_4 x_4 - \beta_5 x_5) - \exp\{-\alpha(y - m - \beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3 - \beta_4 x_4 - \beta_5 x_5\}]$ so that seven parameters $\{\alpha, m, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ were to be estimated. When these were estimated by Newton-Raphson, three samples of the 50 generated had failed to converge by 30 iterations, at which point the calculation was terminated; 5 samples required more than 20 iterations to converge; 4 samples between 11 and 20 iterations; and 38 samples converged after a number of iterations ranging from 6 (the smallest number) to 10. When the 7 parameters were estimated using the IWLS algorithm, with samples generated from the same seed value, the same starting values (all β values equal to zero; m, α equal to their moment estimators) and using the same convergence criteria, convergence was never a problem and was always rapid; 45 samples gave convergence after four iterations, the remainder after three. These results refer to 50 samples of size 50; when 50 samples of size 20 were generated, the superiority of the IWLS was even more apparent. The Newton-Raphson calculation frequently aborted (because of singularity of the matrix of second derivatives of $\log_e L$) and where it did not, many samples showed no convergence by 30 iterations. Using the IWLS algorithm, all 50 samples gave convergence, the number of samples giving convergence after 3, 4 and 5 iterations being 4, 43, and 3. While failure to converge is sometimes encountered when GLMs are fitted by using the IWLS algorithm, it is much less frequent than with Newton-Raphson optimization. Using IWLS, computational difficulties were very occasionally encountered when simulating 200 samples of size $N = 25$, although never with $N = 50$ or $N = 75$.

[19] It is therefore clear that, because the IWLS algorithm exploits the joint linearity of the six parameters $\{m, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$in this example, there is much to be gained by its use. The same is true wherever the trend is of form $\beta^T \boldsymbol{x}$ with coefficients occurring linearly. There are also further advantages to be gained. Observations giving large residuals, or having high leverage, are readily identified; tests of significance (for example, a test of the hypothesis $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ in the example, or that any subset of the β values is zero) are immediate, without the labor of

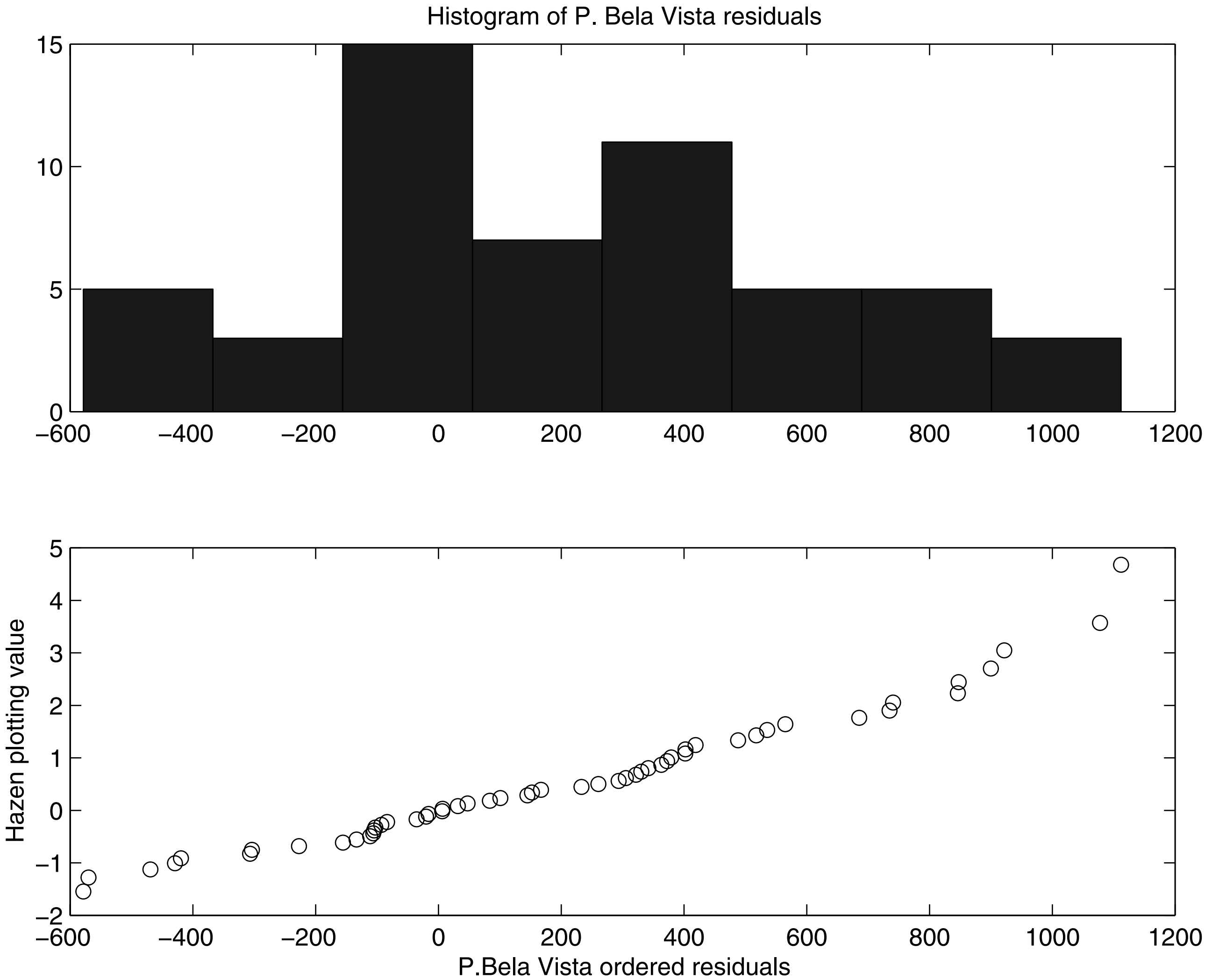

Figure 2. Histogram of Passo Bela Vista residuals and *Q-Q* plot for consistency with the Gumbel hypothesis.

rerunning a Newton-Raphson calculation for the modified model.

7. Comparison of Trends Estimated by Fitting GLMs With Trends Estimated by Regression

[20] The question can also be asked "Why go to the trouble of fitting a GLM where a trend is suspected, instead of simply fitting a trend-line by ordinary linear regression?" One answer, that of logical inconsistency, has already been given. To answer the question further, 200 samples each of two different sizes (N = 25; N = 50) were drawn from a (standard) Gumbel distribution with known parameters (m = 0; α = 1) and linear trends of known magnitude were superimposed on the sample. In the absence of any trend, the variance of the standard Gumbel distribution is $\sigma^2 = \pi^2/6$, so that the standard deviation is σ = 1.2825 approximately; the three positive trends used in the simulation were 0.5σ, σ, and 2σ, distributed over the length N of artificial record. Thus for N = 50 and σ = 1.2825, a linear trend with a single trend coefficient β = 0.02565 was superimposed on the 50 values from the standard Gumbel distribution (50 $\times$ 0.02565 = 1.2825). The three trends β = 0.01282, β = 0.02565 and β = 0.0513 corresponding to 0.5σ, σ and 2σ, were therefore in increasing order of magnitude (but differed for the two cases N = 50 and N = 75 because the latter "record" is longer). Having generated samples with these characteristics, the coefficients of linear trend β were estimated (1) by the GLM procedure described in this paper and (2) by linear regression (LR). The results are shown in Table 3.

[21] While the means of the β values, calculated from the 200 simulated samples, are not far from their true values for both the GLM and LR models, the variances of the 200 β values are substantially different for the two models, the variances of the LR trend parameters being always greater than those given by the GLM procedure. Thus the probabilities $P[\hat{\beta} > 0|H_1]$, where H_1 is the hypothesis that linear trend exists, will be greater for the GLM model than for the LR model.

[22] The next section explores how this last statement can be extended to calculate probabilities in the critical region of a significance test of the hypothesis of zero trend, and hence leads to a conclusion regarding the relative powers of LR

Table 3. Comparison Between Means and Variances of Linear Trend Coefficients β, Calculated by Fitting a GLM to 200 Simulated Samples of Sizes N = 25, 50, and 75 and Means and Variances of Linear Trend Coefficients β, Calculated by Simple Linear Regressions Fitted to the Same 200 Samples[a]

	GLM fitted:	Linear regression fitted:	*var*[LR]/*var*[GLM]
Means of estimated β values.			
N = 25, β = 0.02565	0.0228	0.0258	
N = 25, β = 0.05130	0.0498	0.0531	
N = 25, β = 0.1026	0.1019	0.1020	
N = 50, β = 0.01282	0.0142	0.0136	
N = 50, β = 0.02565	0.0262	0.0253	
N = 50, β = 0.05130	0.0512	0.0504	
N = 75, β = 0.00855	0.0086	0.0085	
N = 75, β = 0.01710	0.0163	0.0164	
N = 75, β = 0.03420	0.0341	0.0341	
Variances of estimated β values (× 10^4)			
N = 25, β = 0.02565	6.8512	10.6755	1.56
N = 25, β = 0.05130	8.6944	18.1912	2.09
N = 25, β = 0.1026	6.9741	9.3634	1.34
N = 50, β = 0.01282	0.8968	1.7144	1.91
N = 50, β = 0.02565	0.9218	1.3429	1.45
N = 50, β = 0.05130	0.8952	1.6691	1.86
N = 75, β = 0.00855	0.3558	0.5488	1.54
N = 75, β = 0.01710	0.2894	0.4033	1.39
N = 75, β = 0.03420	0.3043	0.5072	1.66

[a] Data were sampled from a standard Gumbel distribution (m = 0; α = 1) with added linear trends of magnitude 0.5σ, σ, and 2σ as explained in the text.

and GLM for detecting linear trend, in the particular case where data are simulated from the Gumbel distribution with m = 0, α = 1.

8. Comparison of the Power of Testing for Linear Trend by Linear Regression and GLM Procedure: Data Generated From Gumbel Distributions With *m* = 0, α = 1, and known values of β

[23] To compare the power of the test for linear trend (1) using linear regression (LR) and (2) using the GLM fitting procedure described in this paper, it is first necessary to define an appropriate critical region. For this purpose, the null hypothesis H_0: β = 0 and the unilateral alternative hypothesis H_1: β > 0 were taken, together with a significance probability for the test equal to 5%. To define the critical region appropriate for this test, the distributions of $\tilde{\beta}$ (the slope of a fitted LR) and of $\hat{\beta}$ (the slope estimated by the GLM procedure) were obtained by simulation, using 200 samples of sizes N = 50 and N = 75 generated from the standard Gumbel distribution in which m = 0, α = 1. Figure 3 shows the histograms of $\tilde{\beta}$ and $\hat{\beta}$, under the hypothesis H_0 of trend absent, β = 0; with linear trend present, the histograms are of similar form.

[24] The critical region for the two tests LR, GLM were obtained by calculating the 95% quantiles of these histograms; thus for sample size N = 50, the critical region for LR was $\tilde{\beta}$ > 0.01956 and the critical region for GLM was $\hat{\beta}$ > 0.01707. Thus a proportion 5% of the 200 samples were greater than these values, even though the samples were generated from a trend-free distribution. This gave the type 1 error for the tests equal to 0.05.

[25] The next step was to draw samples of sizes N = 50 and N = 75 from Gumbel distributions with m = 0, α = 1, and various positive values of β, assumed known. The sample sizes 50 and 75 were chosen because pathological samples for which the GLM estimation procedure encountered numerical problems were very unlikely to occur (unlike the case N = 25, where numerical problems, although infrequent, occasionally occurred). For each known value of β, 200 samples were drawn from a Gumbel distribution with m = 0, α = 1, and the appropriate β value giving the trend. From the histograms of the 200 values of $\tilde{\beta}$ (for LR) and $\hat{\beta}$ (for GLM), the proportions of the 200 samples lying in the relevant critical regions were determined, these proportions giving the powers of the LR and GLM tests of the null hypothesis H_0: β = 0 against the one-sided alternative H_1: β > 0. These proportions were equal to 1 minus the type 2 error for the tests. The results are given in Table 4, which shows that the power of the GLM test is greater than the power of the LR test, by a useful margin, the gain in power being greater when the true value β of the trend is not very different from zero. The conclusion is that when data are Gumbel-distributed, testing for trend by the GLM procedure is more sensitive than LR to small departures from the null hypothesis of zero trend; but if the trend is very large, it does not matter which test is used (and, indeed, no formal significance may be required).

9. Discussion

[26] This paper has shown how the very powerful tools of GLM theory can be adapted to test for trend in hydrological variables for which observations consist of their annual maximum values, whether for mean daily discharge, or for rainfall intensity, which as a working hypothesis can be taken to follow a Gumbel distribution with mean that varies over time. If the increases in extreme rainfall and severity of floods come to pass that are predicted as the consequences of climate change, procedures for testing whether and where such changes are occurring will be of great importance.

[27] Whether or not time trends exist, fitting procedures based on GLM theory have wider uses for identifying the relationships between Gumbel-distributed variables, and concomitant variables with which they may be correlated. The example suggested is that GLM procedures could be used to explore how maximum rainfall intensity is related to weather conditions pertaining at the time that the maximum intensities were observed.

[28] An important aspect of GLM theory is the requirement that the variable to be analyzed has a distribution belonging to the exponential family as defined earlier in this paper. Thus GLM procedures can be used to relate data from binomial, Poisson, Normal, gamma and inverse Gaussian distributions to explanatory variables, since these distributions all belong to the exponential family. However the Gumbel distribution does not belong to the exponential family unless its scale parameter is known, so that some ingenuity is required before the power of GLM procedures can be fully exploited in the analysis of Gumbel variates, and for testing hypotheses about their relation to concomitant variables. The paper shows how this can be achieved by what is in effect a two-stage iterative procedure: a first estimate of the scale parameter α is taken, and the remaining parameters m and β (the latter being a vector of trend parameters, or the vector of coefficients of explanatory

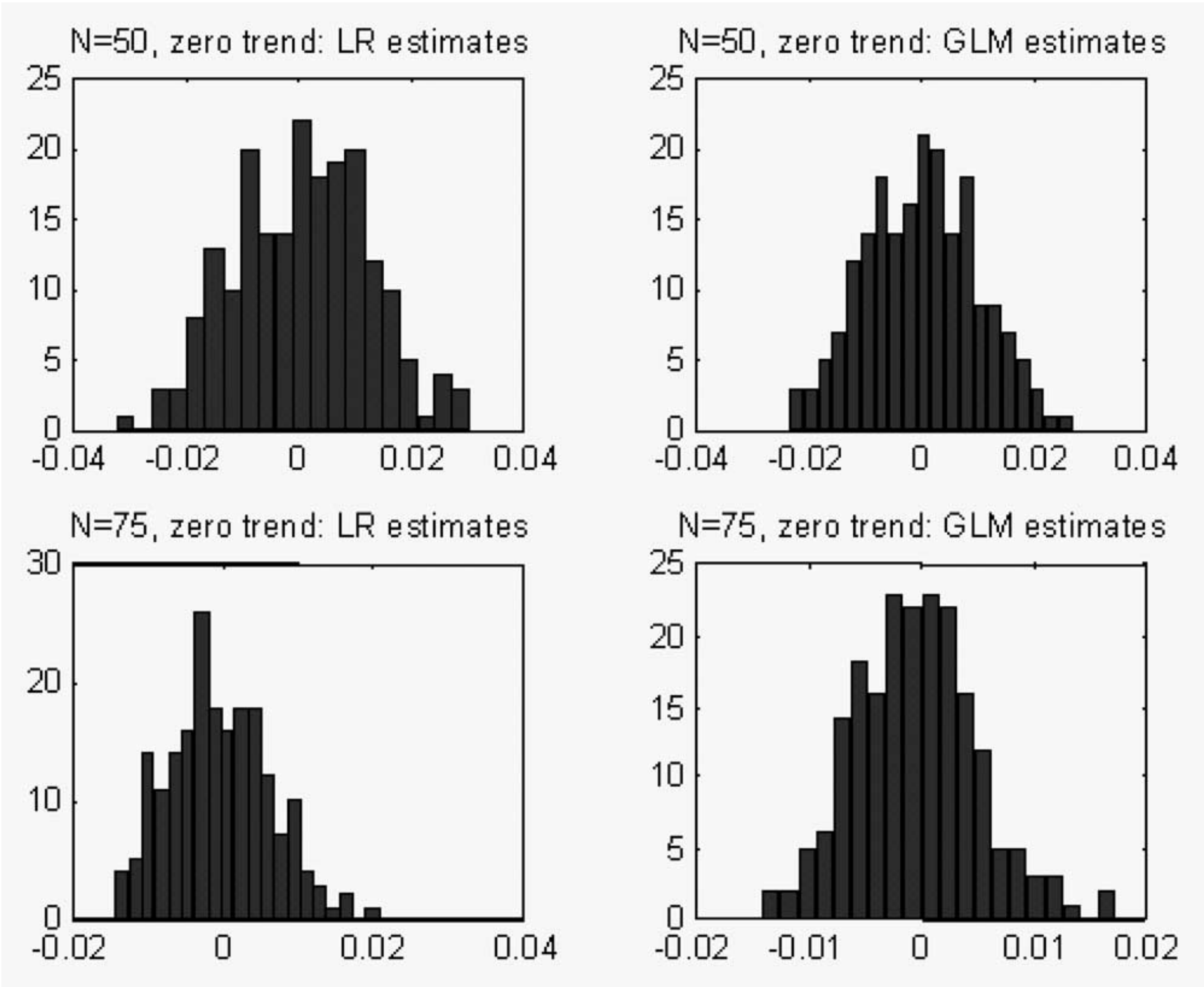

Figure 3. (left) Histograms of 200 linear regression (LR) estimates $\tilde{\beta}$ of linear trend parameter β superimposed on simulated data from a standard Gumbel distribution for sample sizes $N = 50$ and $N = 75$ and (right) histograms of 200 GLM estimates $\hat{\beta}$ for same sample sizes. Horizontal scales differ.

variables) are estimated. With these estimates of m and β, the maximum likelihood equation $\partial \log_e L / \partial \alpha = 0$ is solved iteratively to give an updated α, from which new values of m and β are obtained, the cycle continuing until convergence. In the examples presented in this paper, convergence was extremely rapid. The key to the procedure is to assume certain parameters known so that the distribution of the data becomes a member of the exponential family; and since this procedure is quite general, one may speculate that other distributions used in hydrological practice (such as the Weibull, and 3-parameter gamma) can be treated similarly.

[29] A fairly straightforward extension of the proposed procedure is suggested for testing whether time trends exist in the Gumbel scale parameter α as well as in the Gumbel mean value, but further work is required to explore the convergence properties of this extension. However this paper argues that exploring trends in α is likely to be of secondary interest compared to trends in mean value, because existence of a time trend in the Gumbel mean immediately eliminates the possibility of any analysis based on the assumption of stationary data sequences (such as the calculation of floods with return period T years; or in the case of rainfall intensity data, the calculation of IDF curves). A significant trend in the Gumbel mean shows that more complex methods, using strong assumptions about the future development of the causes of the trend, are required

Table 4. Powers of the LR and GLM Tests for Linear Trend for Two Sample Sizes $N = 50$ and $N = 75$ Calculated From Simulated Samples From a Gumbel Distribution With $m = 0$, $\alpha = 1$ and the Values of β Indicated[a]

	LR	GLM	Ratio, GLM/LR
	N = 50		
$\beta = 0$	0.050	0.050	
$\beta = 0.012825$	0.285	0.345	1.21
$\beta = 0.02565$	0.685	0.800	1.17
$\beta = 0.0513$	1.000	1.000	1
	N = 75		
$\beta = 0$	0.050	0.050	
$\beta = 00855$	0.380	0.480	1.26
$\beta = 0.0171$	0.860	0.940	1.09
$\beta = 0.0342$	1.000	1.000	

[a] Values of β are justified in the text.

if the magnitudes and frequencies of future annual extremes are to be estimated. Whether trends also exist in the Gumbel scale parameter is then of minor interest.

[30] The paper has dealt with the analysis of a single record at one site. In the case of both flood and rainfall intensity data, the hydrologic reality is that records of variable length are available at several sites, and that these are cross-correlated (the correlation between annual maximum mean daily flows at Espumoso and Passo Bela Vista on the Rio Jacuí is evident from inspection of Table 1). Work by *Douglas et al.* [2000] has shown how essential it is to take full account of this correlation when analyzing hydrologic data for the possible existence of time trends. In their analysis of flood records from the Midwest United States, no time trends were detected when spatial correlation was allowed for; but if spatial correlation between records had been neglected, two thirds of the sites analyzed would have shown statistically significant trends. The obvious question therefore is whether the GLM procedure for testing the existence of trend in Gumbel-distributed data can be further extended to incorporate cross-correlation between flood records at different sites. This is a question for future research. However there are hopeful signs that it may be possible. *Clarke* [2001] used GLM procedures to analyze annual flood data in records of variable length and from several sites, assuming that the data were gamma-distributed, with the mean flood in year i at gauge site j expressed in terms of year and site effects a_i, s_j by means of the link-function $\log_e \mu_{ij} = \mu + a_i + s_j$, the component a_i building in a correlation between annual maximum floods recorded in the same year. One may speculate that a similar device might prove useful in the analysis of multisite, Gumbel-distributed data. An obvious difficulty is that the Gumbel scale parameter is likely to vary from site to site, but here GLM theory may give some assistance. Expression (4), giving the general form of the exponential family, includes the function $a(\phi)$. This scaling function is commonly of the form $a(\phi) = \phi/w$ [*McCullagh and Nelder*, 1989] where w is a known prior weight function that varies from observation to observation. In the present context, it may be sufficient to allow w to vary only from site to site, perhaps as a function of drainage basin area, $w = f(A)$. Clearly much more work is required to substantiate such conjectures, or to prove them worthless.

[31] The benefits of casting the trend-detection problem in the form of a GLM are not merely the benefits of computational efficiency and convenience. The simulations described in the paper, using random samples drawn from the standard form of the Gumbel distribution with superimposed linear trends of known magnitude, gave estimates of the trend parameter β by two methods: first, using the GLM procedure explored in the paper, and second, using simple linear regression LR, ignoring the fact that the data have Gumbel properties. Although neither method showed evidence of bias (in terms of deviation from the known value of the trend parameter), the variances of the GLM estimates were always less than the variances of the LR estimates. Translated into terms of the relative power of the two tests, it was shown that when data can be expected to have come from a Gumbel distribution in which a positive linear trend may also be present, the GLM procedure is better able to detect slight departures from the null hypothesis (H_0: $\beta = 0$, trend absent) than simple linear regression.

10. Conclusion

[32] This paper has shown how Gumbel-distributed data can be related to explanatory variables by using Generalized Linear Models fitted by using a modified form of the iteratively weighted least squares algorithm. Typical applications include (1) testing for trend in annual flood data; (2) testing for trend in annual maximum rainfall intensities of different durations, two applications that are relevant in conditions of changing hydrologic regime. Even where trend is absent, the proposed procedure can be used to explore the relation between (say) the observations in a stationary sequence of annual maximum rainfall intensities, and weather variables at the times that the maxima were recorded (such as wind direction, wind velocity). When the Gumbel scale parameter is known, coefficients β of explanatory variables $\boldsymbol{x}$ can be estimated by casting the model in GLM form, and the scale parameter is updated by solution of the maximum likelihood equation for the scale parameter α. The method allows the coefficients of explanatory variables to be estimated rapidly, using computationally efficient methods that take advantage of the linear structure of the parameters m and β in the Gumbel model. Unlike other trend tests (linear regression; or the nonparametric Mann-Kendall) it also utilizes the information that, in the absence of trend, annual floods and annual maximum rainfall intensities can be assumed, as a working hypothesis until disproved, to follow a Gumbel distribution. Computer simulations also showed that, where data have an underlying Gumbel distribution with superimposed linear trend, estimating the trend by means of the GLM procedure is not only more efficient computationally than maximizing the likelihood function by a Newton-Raphson procedure, but also has greater power than linear regression to detect slight departures from the null hypothesis that no trend exists.

[33] **Acknowledgments.** The author is grateful for helpful comments from two anonymous referees and an Associate Editor.

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